NORMAL CURVES ARISING FROM LIGHT OPEN MAPPINGS OF THE ANNULUS(1)

BY
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1. **Introduction.** A mapping δ of an oriented one-dimensional manifold J into the complex plane E^2 is called a *representation*; if δ possesses a continuous nonvanishing tangent δ', then δ is called a regular representation. An image point δ_0 of a regular curve δ is a *vertex* if there exist exactly two distinct points x and y such that $\delta(x) = \delta(y) = \delta_0$ and if the tangents $\delta'(x)$ and $\delta'(y)$ are linearly independent. A regular curve is *normal* if it has a finite number of vertices and every other image point has but one pre-image. Two representations (regular representations) δ and ε are *equivalent* if there exists a sense-preserving homeomorphism $\phi: J \to J$ such that $\varepsilon = \delta \circ \phi$ (and ϕ' is continuous and nonvanishing). A *regular* (*normal*) *curve* is then defined to be an oriented curve with a regular (normal) representation.

Suppose D is an open subset of a two-dimensional manifold and D is bounded by the Jordan curves $J_1, J_2, \cdots, J_n$. Let δ_i be a representation on J_i for $i = 1, \cdots, n$. A continuous function f from $\bar{D}$ into E^2 is called an *extension* of $\delta_1, \cdots, \delta_n$ to D if $f \mid J_i = \delta_i$ for $i = 1, \cdots, n$. Much of the work on extensions has been done for the class of normal curves [5], [9], [10]. A possible reason for this is that well developed combinational tools are available. These tools have been used by Heins and Morse [1], Morse [2], Titus [4], [6] and Titus and Young [7] in their studies of extensions with various analytic or topological properties. In particular Titus [6] has given necessary and sufficient conditions that a normal representation have a light open extension to the disk. The methods developed by Titus are brought to bear in this paper on a related problem; an algorithm is given that yields necessary and sufficient conditions for a pair of normal curves to have a light open extension to the annulus.

2. **Preliminaries.** The notation used is essentially that used in [6]. For convenience a summary of the notation and results of [6] is given in this and the following section.

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In what follows δ will be a representation of a closed curve. Let $[\delta]$ denote the point set consisting of image points of δ. Let $w(\delta, p)$ be the index of δ about a point p not in $[\delta]$.

The *outer boundary* of δ will be the subset of $[\delta]$ which is contained in the closure of the unbounded component of the complement of $[\delta]$; an *outer point* p is a point on the outer boundary such that p has but one pre-image. For normal δ and any nonvertex p one can define $w^+(\delta, p)$ and $w^-(\delta, p)$ as the larger and smaller winding numbers of points p' near p but not on $[\delta]$. An outer point p is *positive* [*negative*] if $w^+(\delta, p) = 1$ [$w^-(\delta, p) = -1$].

Let δ be normal and let $\delta(0)$ be an outer point where δ is given by the complex-valued function $\delta(t) = a(t) + ib(t)$ with t the usual angle parameter, $0 \leqq t < 2\pi$. Index the n vertices in the natural way by traversing the curve with increasing t and using consecutively the integers $0, 1, \cdots, n-1$; thus $\delta_0, \delta_1, \cdots, \delta_{n-1}$ [6, Figure 1, p. 49].

The $2n$ pre-images of the vertices will be denoted by the lower case Roman equivalent of the Greek letter denoting the curve and they will be indexed so that $0 < d_0 < d_1 < \cdots < d_{2n-1} < 2\pi$. Denote d_j also by d_k^* if $\delta(d_j) = \delta(d_k)$ for $j \neq k$; thus, $\delta(d_k^*) = \delta(d_k)$ for all k. Define the function v by

$$v(d_k) = v(k) = \operatorname{sgn} \begin{vmatrix} a'(d_k^*) & b'(d_k^*) \\ a'(d_k) & b'(d_k) \end{vmatrix}.$$

For a normal δ with $p = \delta(0)$ a positive outer point, the *intersection sequence* of δ with respect to p is defined by the sequence $\{d_k\}$, the values of d_k^* and the values of $v(k)$ for each k. A pair of normal representations δ and ε have *isomorphic intersection sequences* if they have the same number of vertices, $d_j^* = d_k$ if $e_j^* = e_k$, and $v(d_k) = v(e_k)$. If two representations have isomorphic intersection sequences and one has an extension with certain topological properties, so will the other, as can be seen from the following theorem:

THEOREM [6, p. 49]. *Let δ be a normal representation defined on a Jordan curve D; let ε be normal and defined on the Jordan curve E. Suppose that each curve has a positive outer point and the curves have isomorphic intersection sequences. Then there exists a sense-preserving homeomorphism h of E^2 onto E^2 taking D onto E such that $\varepsilon h = \delta$ (juxtaposition denotes function composition).*

Thus, for our purposes two representations with isomorphic intersection sequences are interchangeable.

3. **Interior boundaries.** Let D be an open subset of a two-dimensional manifold bounded by a finite number of Jordan curves and let f be a mapping of $\bar{D}$ into E^2. If f is light, open, and sense-preserving on D, and a local homeomorphism (relative to $\bar{D}$) at each point of Bdry D, then f is called *properly interior*. We shall have use for a theorem of Stoïlow [3], [11, p. 88] which states that if f is properly

interior on $\bar{D}$, then at each point p of D there exists a closed two-cell neighborhood of p on which f is topologically equivalent to $w = z^n$ on the unit disk for some positive integer n.

A regular representation which has a properly interior extension to the disk is an *interior boundary*. The necessary and sufficient conditions given in [6] that a normal curve be an interior boundary are outlined in this section.

We have required that interior boundaries have sense-preserving light open extensions. It is well known that such representations have non-negative circulation; hence, the "cut" process described below only considers such representations. It will be necessary in §5 to use the "cut" process for representations with non-positive circulation; however, it will be apparent what adjustments should be made.

Suppose then that δ is a normal representation of a closed curve with non-negative circulation. For such representations, $v(d_0) = 1$. Since $v(d_0^*) = -v(d_0) = -1$, there is some index j such that $v(j) = -1$; let k be the smallest such index. One then has the following cases: (I) $d_k^* < d_k$, (II) $d_k < d_k^*$; in the latter case one has for each $j < k$ the subcases: (II′(j)) $d_j < d_k < d_k^* < d_j^*$, (II″(j)) $d_j < d_k < d_j^* < d_k^*$.

For each k and j chosen as above (j is chosen only in case II) a "cut" will be defined. Each "cut" will lead to a pair of normal representations to be used as criteria for deciding if δ is an interior boundary.

Suppose δ and ε are representations and $x = \delta(p)$ and $y = \delta(q)$ are two points such that $p < q$. Denote by $\delta(p)\ \delta(q)$ the representation obtained by restricting δ to $p \leqq t \leqq q$. If x and y each have only one pre-image, then $xy(\delta)$ or xy will also be used to denote this representation. Let $-\delta$ be the representation gotten by tracing δ in the opposite direction. Consistent with this notation $\delta(q)\delta(p)(-\delta)$ traces from $\delta(q)$ to $\delta(p)$ via $-\delta$. We shall use $ap(\delta) + pb(\varepsilon)$ to denote the representation $ap(\delta)$ followed continuously by $pb(\varepsilon)$, where p is in $[\delta]$ and $[\varepsilon]$.

Given normal δ define the representations δ^* and δ^{**} as follows (see Figures 2, 3, 4 of [6, pp. 51–52]):

Case I

$$\delta^* = \delta(d_k^*)\delta(d_k),$$

$$\delta^{**} = \delta(0)\delta(d_k^*) + \delta(d_k)\delta(2\pi);$$

Case II′(j).

$$\delta^* = \delta(d_j)\delta(d_k) + \delta(d_k^*)\delta(d_j^*),$$

$$\delta^{**} = \delta(0)\delta(d_k^*) + \delta(d_k)\delta(d_j)(-\delta) + \delta(d_j^*)\delta(2\pi);$$

Case II″(j). δ^*

$$= \delta(d_j^*)\delta(d_k^*) + \delta(d_k)\delta(d_j)(-\delta),$$

$$\delta^{**} = \delta(0)\delta(d_j^*) + \delta(d_j)\delta(d_k) + \delta(d_k^*)\delta(2\pi).$$

In all cases δ is an interior boundary if and only if δ^* and δ^{**} are. Since δ^* and δ^{**} are not normal, they are modified to normal representations $\bmod \delta^*$ and $\bmod \delta^{**}$ [6, p. 55 and p. 57]. The modifications are done so that δ is an interior boundary if and only if $\bmod \delta^*$ and $\bmod \delta^{**}$ are interior boundaries, $\bmod \delta^*$ and $\bmod \delta^{**}$ are normal and $\bmod \delta^*$ and $\bmod \delta^{**}$ have strictly less vertices than δ. In a finite number of steps the representation δ can be "cut" into Jordan curves so that δ is an interior boundary if and only if all the Jordan curves are interior boundaries, i.e., positively oriented.

4. **An arc-lifting theorem.** Theorem 1 will be a useful tool in the next section. In this section a *Jordan curve* J on an orientable two-dimensional manifold will be an oriented simple closed curve that bounds a two-cell. The two-cell bounded by J will be denoted by $\operatorname{Ins} J$. Jordan curves will be ordered by picking a fixed starting point and ordering the curve by positive traversal.

THEOREM 1. *Let D be an open subset of an orientable two-dimensional manifold M such that $\bar{D}$ is compact and let J be a Jordan curve on M. Suppose D is a subset of* $\operatorname{Ins} J(M - \operatorname{Ins} J)$ *such that J is a component of* Bdry D. *Let f be a properly interior mapping of $\bar{D}$ into E^2 and let $f \mid J = \delta = u + iv$ be a regular representation. Suppose $\varepsilon(t) = a(t) + ib(t)$, $0 \leqq t \leqq 2\pi$, is a regular representation of an arc in E^2 with $y = \varepsilon(q)$ for $0 < q < 2\pi$. If either* (1) *$y = \delta(p)$ for some p in J and*

$$\operatorname{sgn} \begin{vmatrix} u'(p) & v'(p) \\ a'(q) & b'(q) \end{vmatrix} = -1\,(+1)$$

or (2) *there is a point p in D such that $f(p) = y$, then there exists an arc B in $\bar{D}$ with end points p and b such that $B - \{p, b\} \subset D$ and such that f maps B homeomorphically into $[\varepsilon]$. In either case, if b is in D, then $f(B) = [\varepsilon(0)\,\varepsilon(q)]$.*

Proof. Suppose (1) holds. Select an open set U in D such that $f \mid \bar{U}$ is a homeomorphism, $\bar{U} - U$ is a Jordan curve, and $J \cap (\bar{U} - U)$ is an arc containing p in its interior. If J is given a positive orientation, then $\bar{U} - U$ is positively (negatively) oriented since U is contained in $\operatorname{Ins} J(M - \operatorname{Ins} J)$. Because f is a sense-preserving homeomorphism, $f(\bar{U} - U)$ is a positively (negatively) oriented Jordan curve. Choose r and s so that $0 < r < q < s$ and $[\varepsilon(r)\varepsilon(s)] \cap f(\bar{U} - U) = \{y\}$. Let x be any point with $r \leqq x < q$. From [5, Lemma 2, p. 1085] we have that

$$w(f \mid \bar{U} - U, \varepsilon(s)) - w(f \mid \bar{U} - U, \varepsilon(x)) = \operatorname{sgn} \begin{vmatrix} u'(p) & v'(p) \\ a'(q) & b'(q) \end{vmatrix}.$$

Because of (1), the last term is $-1\,(+1)$. Since f describes a positively (negatively) oriented Jordan curve, the only possible values of the index are $+1\,(-1)$ and 0; consequently, $w(f \mid \bar{U} - U, \varepsilon(x)) = 1\ (-1)$. This can only happen if $\varepsilon(x)$ is in $f(U)$; thus, $P = \{\varepsilon(t) \mid r \leqq t < q\}$ is contained in $f(U)$. Since f is a homeomorphism on $\bar{U}$, there is an arc Q in $\bar{U}$ with end point p mapping homeomorphically onto P and $Q \cap (\bar{U} - U) = \{p\}$.

Let $P' = [\varepsilon(0)\varepsilon(r)]$ and suppose K is the component (in D) of $f^{-1}(P')$ containing a, the other end point of Q. Such a component is nondegenerate since f is locally z^n at a. The set $f(\bar{K})$ is an arc as it is a connected subset of $[\varepsilon]$; thus $f(\bar{K}) = [\varepsilon(z)\varepsilon(r)]$ for some z. Let b be a pre-image of $\varepsilon(z)$ in $\bar{K}$. Since $K \cup \{b\}$ is a nondegenerate connected set, there is an arc K' in K with end points a and b. If b is not in $\bar{D} - D$ and $f(b) \neq \varepsilon(0)$, there is an arc A at b mapping into $[\varepsilon(0)\varepsilon(z)]$ since f is locally z^n at b. Now K is a component so A must be contained in K; this is impossible since $f(K) \cap [\varepsilon(0)\varepsilon(z)] = \varnothing$. Therefore, if b is not in $\bar{D} - D$, $f(b) = \varepsilon(0)$ and $f(Q \cup K') = [\varepsilon(0)\varepsilon(q)]$. Note that f is homeomorphic on $Q \cup K'$ [11, Theorem 4.1, p. 96]; hence $Q \cup K'$ is the desired arc.

If (2) holds, take K to be the component (in D) of $f^{-1}([\varepsilon(0)\varepsilon(q)])$ containing p. An argument similar to that of the above paragraph produces the desired arc.

5. **Interior mappings on the annulus.** Let A denote an open annulus in the plane bounded by Jordan curves C_1 and C_2, where C_1 is contained in Ins C_2. If δ and ε are regular representations, we say (δ, ε) is an *a-boundary* when there exists a properly interior f on $\bar{A}$ such that $f \mid C_2 = \delta$ and $f \mid C_1 = \varepsilon$.

LEMMA 5.1. *Suppose δ is an interior boundary defined on a positively oriented Jordan curve J. Let $J = T_1 \cup T_2 \cup T_3 \cup T_4$, where the T_i are arcs which only intersect at the end points and the T_i are numbered as J is traversed in the positive order. If $\delta \mid T_3 = -(\delta \mid T_1)$, then $(\delta \mid T_2, -(\delta \mid T_4))$ is an a-boundary.*

Proof. Suppose without loss of generality that $J = \text{Bdry}\{z \mid 1 \leqq |z| \leqq 2, 0 \leqq \arg z \leqq \pi\}$ and $D = \text{Ins} J$. Let f be a properly interior extension of δ on $\bar{D}$. It can also be assumed that $T_1 = \{z \text{ in } J \mid z \text{ real}, 1 \leqq z \leqq 2\}$, $T_2 = \{z \text{ in } J \mid |z| = 2\}$, $T_3 = \{z \text{ in } J \mid z \text{ real}, -2 \leqq z \leqq -1\}$, and $T_4 = \{z \text{ in } J \mid |z| = 1\}$; also one can assume, in view of the hypothesis, that $f(x) = f(-x)$ for real x in J.

Let $g(z) = z^2$ for z in $\bar{D}$. Define h to be fg^{-1}; note h is well-defined and continuous on the annulus $\bar{A} = \{z \mid 1 \leqq |z| \leqq 4\}$. Clearly h is open at each point of A except possibly at the real positive points of A; therefore, h is open on A [8, Theorem 9, p. 336]. The mapping h is the desired extension of $(\delta \mid T_2, -(\delta \mid T_4))$.

DEFINITION. Two normal representations δ and ε *intersect normally* if $[\delta] \cap [\varepsilon]$ is a finite set, if no point of $[\delta] \cap [\varepsilon]$ is a vertex of either curve, and if the tangents to the curves at each point of intersection are linearly independent.

THEOREM 2. *Let δ and ε be normal representations of closed curves which intersect normally. Suppose $-\varepsilon$ is not an interior boundary. Then (δ, ε) is an a-boundary if and only if one of the following holds:*

(1) *Suppose ε has some points of positive circulation. Let p be a point not in $[\varepsilon]$ such that $w(\varepsilon, p) > 0$. Suppose ϕ represents an arc which intersects δ and ε normally and which has one end point at p and the other at a point q in the*

unbounded component of $E^2 - [\delta] - [\varepsilon]$. *If* $[\phi] \cap [\delta] = \{a_1, \cdots, a_m\}$ *and* $[\phi] \cap [\varepsilon] = \{b_1, \cdots, b_n\}$, *then the curve*

$$\zeta^{ij} = \delta(0)a_i + a_i b_j(\phi) + b_j b_j(-\varepsilon) + b_j a_i(-\phi) + a_i \delta(0)$$

is an interior boundary for some i and j, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$.

(2) *Suppose* ε *has nonpositive circulation and* ε *has a cut of Type* I *at* $\varepsilon_r = \varepsilon(e_k)$ *with* $e_k^* < e_k$. *Then either*

(a) *there is a point p in* $[\delta] \cap [\varepsilon(e_k^*)\varepsilon(e_k)]$ *such that* ζ^n *is an interior boundary for some n*, $0 \leqq n \leqq w^+(\delta, p) + w^+(-\varepsilon, p)$, *where* $\zeta^0 = \delta(0)p + p\varepsilon(e_k) + \varepsilon(e_k)\varepsilon(e_k)(-\varepsilon) + \varepsilon(e_k)p(-\varepsilon) + p\delta(0)$ *and* $\zeta^n = \delta(0)p + p\varepsilon(\varepsilon_k) + \sum_{i=1}^n \varepsilon(e_k^*)\varepsilon(e_k) + \varepsilon(e_k)\varepsilon(e_k)(-\varepsilon) + \sum_{i=1}^n \varepsilon(e_k)\varepsilon(e_k^*)(-\varepsilon) + \varepsilon(e_k)p(-\varepsilon) + p\delta(0)$.

(b) ε^* *represents a negatively oriented Jordan curve and* $(\delta, \varepsilon^{**})$ *is an a-boundary.*

(3) *Suppose* ε *has nonpositive circulation and* ε *has a cut of Type* II *at* $\varepsilon_r = \varepsilon(e_k)$ *with* $e_k < e_k^*$. *Then either*

(a) *there exists a point p in* $[\delta] \cap [\varepsilon(0)\varepsilon(e_k)]$ *such that* $\zeta = \delta(0)p + p\varepsilon(e_k) + \varepsilon_r\varepsilon_r(-\varepsilon) + \varepsilon(e_k)p(-\varepsilon) + p\delta(0)$ *is an interior boundary*;

(b) *if* ϕ *represents an arc with end point* $\varepsilon(0)$, *in the unbounded component of* $E^2 - [\varepsilon]$, *and such that* ϕ *and* δ *intersect normally at* $\{a_1, \cdots, a_m\}$, *then the curve* $\zeta^i = \delta(0)a_i + a_i\varepsilon(0)(\phi) + \varepsilon(0)\varepsilon(e_k) + \varepsilon_r\varepsilon_r(-\varepsilon) + \varepsilon(e_k)\varepsilon(0)(-\varepsilon) + \varepsilon(0)a_i(-\phi) + a_i\delta(0)$ *is an interior boundary for some i*, $1 \leqq i \leqq m$;

(c) $-\varepsilon^*$ *is an interior boundary and* $(\delta, \varepsilon^{**})$ *is an a-boundary*;

(d) $-\varepsilon^{**}$ *is an interior boundary and* (δ, ε^*) *is an a-boundary.*

Necessity proof. Let f be a properly interior extension of (δ, ε) on $\bar{A}$.

Suppose ε has some points of positive circulation and let p, q, the a_i, and the b_i be as in (1) of the theorem. Let $\varepsilon = u(t) + iv(t)$ and $\phi = a(t) + ib(t)$ be parametrized on C_1 so that the pre-image of b_i under ε and ϕ is t_i, $1 \leqq i \leqq n$. Define σ_i by

$$\sigma_i = \operatorname{sgn} \begin{vmatrix} u'(t_i) & v'(t_i) \\ a'(t_i) & b'(t_i) \end{vmatrix}.$$

Let ϕ be ordered as it increases from p to q; assume $b_i \leqq b_j$ if and only if $i \leqq j$. Since q is in the unbounded component of $E^2 - [\delta] - [\varepsilon]$, $w(\varepsilon, q) = 0$; thus $\sum_{i=1}^n \sigma_i = w(\varepsilon, p)$ [5, Lemma 2, p. 1085]. By hypothesis, this last quantity is strictly positive. For each j such that $\sigma_j = 1$, Theorem 1 can be applied at t_j, yielding an arc B_j with end points t_j and x_j and which maps into $[b_j q(\phi)]$. If x_j were in A, then x_j would map onto q. This is not possible, since no point of A can map into the unbounded component of $E^2 - [\delta] - [\varepsilon]$ under a properly interior mapping; thus, x_j is in C_1 or C_2. If x_j is in C_1, then $x_j = t_k$ for some k and $\sigma_k = -1$. If every x_j is in C_1, then there are at least as many $\sigma_k = -1$ as $\sigma_j = 1$. This contradicts the fact that $\sum_{i=1}^n \sigma_i > 0$; hence, some x_j is in C_2 and

$f(x_j) = a_i$ for some i, $1 \leqq i \leqq m$. Suppose D and g are defined as in the proof of Lemma 5.1 and $W = \{z \mid 1 < |z| < 4\}$. By composing f with an appropriate homeomorphism of $\bar{W}$ onto the domain of f, we may assume that the domain of f is $\bar{W}$ and $B_j = \{z \mid 1 < |z| < 4, \arg z = 0\}$. The mapping fg is properly interior on $\bar{D}$ and extends the curve ζ^{ij} of (1).

Suppose ε has nonpositive circulation and ε has a cut of Type I at some $\varepsilon_r = \varepsilon(e_k)$. By definition of a Type I cut, $v(e_k) = +1$ with $e_k^* < e_k$. Recall that $\varepsilon^* = \varepsilon(e_k^*)\varepsilon(e_k)$; ε^* represents a negatively oriented Jordan curve [6, Lemma 5, p. 53]. Choose p on $[\varepsilon^*]$ with $p \neq e_r$. Since $v(e_k) = +1$, by Theorem 1 there is an arc L_1' with end point at e_k^* which maps into $[p\varepsilon(e_k)(\varepsilon)]$. If the other end point z of L_1' is not in $\bar{A} - A$, then $f(z) = \varnothing$. Again by Theorem 1, there must be an arc L_1'' with end point at z which maps into $[\varepsilon(e_k^*)p(\varepsilon)]$. Let $L_1 = L_1' \cup L_1''$. If the other end point x of L_1 is in A, we apply this process again, obtaining an arc L_2 with one end point at x and which maps into $[\varepsilon^*]$. The process must terminate after a finite number of steps since properly interior mappings are finite-to-one. Thus there exists an arc $K = L_1 \cup L_2 \cup \cdots \cup L_s$ where each L_i maps onto the Jordan curve determined by ε^* for $1 \leqq i < s$. One end point of K is e_k^*; the other, y, is either in C_1 or C_2.

If y is in C_2, define W, D, and g as before. Assume the domain of f is $\bar{W}$ and $K = \{z \mid 1 < |z| < 4, \arg z = 0\}$. Then fg is a properly interior extension of ζ^{s-1} in (2a). For each L_i, there must be points near L_i mapping onto points near $p = f(y)$. The number of pre-images of a point p' not in $[\delta]$ or $[\varepsilon]$ is $w(\delta, p') + w(-\varepsilon, p')$ [2, p. 72]. If p' is near p, then $w(\delta, p') \leqq w^+(\delta, p)$ and $w(-\varepsilon, p') \leqq w^+(-\varepsilon, p)$. Thus $s - 1 \leqq w^+(\delta, p) + w^+(-\varepsilon, p)$.

If y is in C_1, $y = e_k$ since f is a local homeomorphism on C_1. Let J be the positively oriented Jordan curve determined by K and $V = \{t \text{ in } C_1 \mid e_k^* \leqq t \leqq e_k\}$ and let J' be the positively oriented Jordan curve determined by K and $C_1 - V$. Either $\operatorname{Ins} J$ is contained in A or $\operatorname{Ins} J$ contains the points of $C_1 - V$. Suppose by way of contradiction that the latter case occurred. Let T be the unit circle, let $U_1 = \{z \text{ in } \operatorname{Ins} T \mid \operatorname{Im} z > 0\}$, and let $U_2 = \{z \text{ in } \operatorname{Ins} T \mid \operatorname{Im} z < 0\}$. There exists a homeomorphism h on $\bar{U}_2 \cup T$ such that $h(T) = -C_1$, $h(\operatorname{Bdry} U_1) = -J$, $h(U_2) = \operatorname{Ins} J'$, and h is sense-preserving on U_2. The mapping fh is light open on U_2. Since f maps J onto a negatively oriented Jordan curve, fh maps Bdry U_1 onto a positively oriented Jordan curve; thus, on Bdry U_1, fh is topologically equivalent to $w = z^{s+1}$ [11, Theorem 4.3, p. 86]. Define fh to be topologically equivalent to $w = z^{s+1}$ on $\bar{U}_1$; then fh is light open on $\operatorname{Ins} T$ [8, Theorem 9, p. 336]. Since $fh \mid T = f \mid -C_1 = -\varepsilon$, the curve $-\varepsilon$ is an interior boundary. This is contrary to hypothesis; thus $\operatorname{Ins} J$ is contained in A.

On J the mapping f is topologically equivalent to $w = z^{s+1}$ on the unit circle [11, Theorem 4.3, p. 86]; hence, f can be defined on $J \cup \operatorname{Ins} J$ to be topologically equivalent to $w = z^{s+1}$ on the unit disk. Thus there are arcs X and Y in $\operatorname{Ins} J$ such that X has end point e_k^*, Y has end point e_k, X and Y intersect only at the other end point, f is a homeomorphism on X and on Y, and $f(X) = f(Y)$. Let A_1

be the open annulus bounded by C_2 and by the Jordan curve determined by X, Y, and $C_1 - V$. There exists a map h from $\bar{A}_1$ onto $\{z \mid 1 \leqq |z| \leqq 2\}$ such that h is a homeomorphism on $\bar{A}_1 - X - Y$, on X, and on Y. Also $h(X) = h(Y) = \{z \mid 1 \leqq |z| \leqq 3/2, \arg z = 0\}$ and, for x in X and y in Y, $h(x) = h(y)$ if and only if $f(x) = f(y)$. Clearly fh^{-1} is well-defined, continuous everywhere, light, and open except possibly at $h(X)$; therefore, fh^{-1} is light open on $h(X)$ [8, Theorem 9, p. 336]. The mapping fh^{-1} is a properly interior extension of $(\delta, \varepsilon^{**})$; thus $(\delta, \varepsilon^{**})$ is an a-boundary. Since ε^* describes a negatively oriented Jordan curve, $-\varepsilon^*$ is an interior boundary. This gives case (2b).

Suppose ε has nonpositive circulation and ε has a cut of Type II at some $\varepsilon_r = \varepsilon(e_k)$. Select k to be the smallest integer such that $v(e_k) = 1$. Let ϕ be a representation as described in (3b) of the theorem; let $V = [\phi] \cup [\varepsilon(0)\varepsilon(e_k)(\varepsilon)]$. Since $v(e_k) = 1$, it follows from Theorem 1 that there is an arc K with end point e_k^* mapping into V. If the other end point x of K is in A, K maps onto V; however, this is not possible since a properly interior mapping cannot map points of A into the unbounded component of $E^2 - [\delta] - [\varepsilon]$. Hence, x is in C_1 or C_2.

If x is in C_2, define W, D, and g as before. Once again we may assume the domain of f is $\bar{W}$ and $K = \{z \mid 1 < |z| < 4, \arg z = 0\}$. If $f(x) = p$ is a point of $[\varepsilon(0)\varepsilon(e_k)]$, then fg is a properly interior extension on $\bar{D}$ of ζ in (3a). If $f(x) = a_i$ is a point of $[\phi]$, then fg is a properly interior extension on $\bar{D}$ of ζ^i in (3b).

Suppose x is in C_1. Since f is a homeomorphism at x, $f(x)$ must be a vertex ε_p. By definition of V, $p \leqq r$; however, for Type II cuts it must be that $p < r$. Let $\varepsilon_p = \varepsilon(e_j)$ with $e_j < e_k < e_k^*$. Note that $x = e_j^*$. Assume that $e_k^* < e_j^*$; the proof for $e_j^* < e_k^*$ is similar. Let J be the Jordan curve determined by K and $\{t \mid e_k^* \leqq t \leqq e_j^*\}$ and let L be the Jordan curve determined by K and $C_1 - J$. Orient these curves by the orientation of C_1. By definition of Type II cuts, $f \mid J = \varepsilon^*$ and $f \mid L = \varepsilon^{**}$. Either $\operatorname{Ins} J$ or $\operatorname{Ins} L$ is a disk contained in A. If $\operatorname{Ins} J$ is a disk contained in L, the restriction of f to $\operatorname{Ins} J$ gives a light open extension of ε^*. But J is negatively oriented; hence, $-\varepsilon^*$ is an interior boundary. The restriction of f to the annulus bounded by C_2 and the positively oriented Jordan curve L extends $(\delta, \varepsilon^{**})$; thus, $(\delta, \varepsilon^{**})$ is an a-boundary. This is case (3c). If $\operatorname{Ins} L$ is a disc contained in A, a similar argument shows that $-\varepsilon^{**}$ is an interior boundary and (δ, ε^*) is an a-boundary. This gives case (3d), completing the necessity proof.

Sufficiency proof. If condition (1), (2a), (3a), or (3b) holds, it follows from Lemma 5.1 that (δ, ε) is an a-boundary.

Suppose that (2b) holds where ε has a cut of Type I. Then $(\delta, \varepsilon^{**})$ is an a-boundary; also, ε^* describes a negatively oriented Jordan curve. Number the vertices of ε^{**} as if they were vertices of ε. The point ε_r is not a vertex of ε^{**}. Recall that $[\varepsilon^*]$ intersects $[\varepsilon^{**}]$ only in the point ε_r [6, Lemma 5, p. 53].

Let f be a properly interior extension of $(\delta, \varepsilon^{**})$ on $\bar{A}$. Choose an arc B in $\operatorname{Ins}[\varepsilon^*]$ with end point at ε_r. By Theorem 1 there is an arc K in A with end point on C_1 mapping homeomorphically onto B. Let $A_1 = \{z \mid 1 < |z| < 2\}$, $X = \{z \mid |z| = 1, \, ^{-}\pi/2 \leqq \arg z \leqq 0\}$, and $Y = \{z \mid |z| = 1, \, 0 \leqq \arg z \leqq \pi/2\}$. There exists a mapping h_1 from $\bar{A}_1$ to $\bar{A}$ such that h is a homeomorphism on $\bar{A}_1 - X - Y$, on X, and on Y; also, $h_1(X) = h_1(Y) = K$. Let $L = \{z \mid 0 \leqq |z| \leqq 1$, z imaginary$\}$ and let U be the domain bounded by X, Y, and L. There exists a mapping h_2 properly interior on $\bar{U}$, except at $z = 1$, which maps L onto $[\varepsilon^*]$ and such that $h_2 \mid X = fh_1 \mid X$ and $h_2 \mid Y = fh_2 \mid Y$. Define h on $\bar{A}_1 \cup \bar{U}$ by $h \mid \bar{U} = h_2$, $h \mid \bar{A}_1 = fh_1$. Then h is a properly interior extension of (δ, ε).

Suppose that (3c) holds and ε has a cut of Type II′ (j) or Type II″ (j) at ε_r. The arc $T = [\varepsilon(e_j)\varepsilon_r(\varepsilon)]$ is traced in opposite directions by ε^* and ε^{**}; hence, in the same direction by $-\varepsilon^*$ and ε^{**}.

Let A_1, X, Y, and U be defined as before. Choose f a properly interior extension of $(\delta, \varepsilon^{**})$ on $\bar{A}_1$. Let $V = X \cup Y$ and suppose without loss of generality that V is the arc mapped onto T by f. Since $-\varepsilon^*$ is an interior boundary which traces T in the same direction as $f \mid V$, there exists a properly interior extension g of $-\varepsilon^*$ on U such that V is mapped onto T. Define h on $\bar{A}_1 \cup \bar{U}$ by $h \mid \bar{A}_1 = f$ and $h \mid \bar{U} = g$. Then h is properly interior on $\bar{A}_1 \cup \bar{U}$ and extends (δ, ε).

The proof for (3d) is similar.

This completes the proof of the theorem.

Definition. Let δ be a representation of a closed curve. An arc B is an interior arc of δ with end point p if p is one end point of B, $[\delta] \cap B = \{p\}$, and $w(\delta, p') = w^+(\delta, p)$ for all p' in B.

Lemma 5.2. *Suppose δ and ε are normal interior boundaries with p a point of $[\delta]$ and q a point of $[\varepsilon]$, where neither p nor q is a vertex. If ϕ represents an arc B from p to r such that B is an interior arc of δ at p and $[qr(\phi)]$ is an interior arc of ε at q, then $(\delta, -\varepsilon)$ is an a-boundary.*

Proof. Let D be a disk divided into disks G and H by an arc E. From [6, Lemma 6, p. 53] we see that $\delta' = \delta(0)p + \phi + (-\phi) + p\delta(0)$ has extension g on $\bar{G}$ and $\varepsilon' = \varepsilon(0)q + qr(\phi) + rq(-\phi) + q\varepsilon(0)$ has extension h on $\bar{H}$ such that g and h are light open and properly interior except at the points of Bdry G and Bdry H mapping onto r. Without loss of generality assume that $h \mid E = g \mid E = qr(\phi) + rq(-\phi)$. There must be arcs U and V on Bdry G such that $g \mid U = pq(\phi)$ and $g \mid V = qp(-\phi)$. Define f on $\bar{D}$ by $f \mid \bar{G} = g$ and $f \mid \bar{H} = h$; then f is properly interior on $\bar{D}$ [8, Theorem 9, p. 336]. The conclusion follows from an application of Lemma 5.1 to $f \mid$ Bdry D.

Lemma 5.3. *If (δ, ε) is an a-boundary, then so is $(-\varepsilon, -\delta)$.*

Proof. Choose f a properly interior extension of (δ, ε) on A. There exists a homeomorphism h on A topologically equivalent to $1/z$ on $\{z \mid 1 \leqq z \leqq 2\}$ such that $h \mid C_1 = -C_2$ and $h \mid C_2 = -C_1$. The mapping fh extends $(-\varepsilon, -\delta)$.

THEOREM 3. *Let δ and ε be normal interior boundaries which intersect normally.*

(1) *Suppose $[\delta] \cap [\varepsilon] = \varnothing$. Then $(\delta, -\varepsilon)$ is an a-boundary if and only if $[\varepsilon]$ is contained in a component P of $E^2 - [\delta]$ such that $w(\delta, p) > 0$ for p in P or $[\delta]$ is contained in a component Q of $E^2 - [\varepsilon]$ such that $w(\varepsilon, q) > 0$ for q in Q.*

(2) *If $[\delta] \cap [\varepsilon] \neq \varnothing$, then $(\delta, -\varepsilon)$ is an a-boundary.*

Proof. *Case* 1. Suppose $(\delta, -\varepsilon)$ is an a-boundary. Either $[\delta]$ is contained in U, the unbounded component of $E^2 - [\varepsilon]$, or $[\varepsilon]$ is contained in the unbounded component of $E^2 - [\delta]$ (or both). Suppose the former holds. Let J be a Jordan curve in U such that $[\varepsilon]$ is in Ins J and $[\delta]$ is in the other component of $E^2 - J$. Now there must be a point y of $f(A)$ on J; otherwise, $f(A)$ would not be connected. Thus the number of pre-images of y in $A, n(y)$, is strictly positive. By [2, p. 72], $n(y) = w(\delta, y) + w(-\varepsilon, y)$ and since y is in U, $w(-\varepsilon, y) = 0$. Index is constant on components; thus, $w(\delta, p) = w(\delta, y) = w(\delta, y) + w(-\varepsilon, y) = n(y) > 0$ for any p in the component P of $E^2 - [\delta]$ containing y. Since $[\varepsilon]$ is contained in P the conclusion follows. The proof for the case where $[\varepsilon]$ is contained in the unbounded component of $E^2 - [\delta]$ is similar.

Now suppose $[\varepsilon]$ is in a component P of $E^2 - [\delta]$ such that $w(\delta, p) > 0$ for p in P. Let ϕ represent an arc with one end point at r in P, the other at s in the unbounded component of $E^2 - [\delta]$, such that ϕ and δ intersect normally, $[\phi] \cap [\varepsilon] = \{q\}$, and $qr(-\phi)$ is an interior arc of ε at q. Let h be a properly interior extension of δ on the unit disk D. Since $w(\delta, r) > 0$, r has a pre-image x in D [2, p. 72]. By Theorem 1, there is an arc K with end point x mapping into $[\phi]$. The other end point y of K must be in Bdry D since K cannot map onto $[\phi]$. Let $p = h(y)$ and $\zeta = pr(-\phi)$. Without loss of generality we may assume that $K = \{z \mid 0 < |z| < 1, \ \arg z = 0\}$. Let $W = \{z \mid |z| < 1, \ \operatorname{Im} z > 0\}$ and $g(z) = z^2$ for z in $\bar{W}$. Then hg is a light open extension of $\delta(0)p + \zeta + (-\zeta) + p\delta(0)$. The proof now proceeds exactly as the proof of Lemma 5.2, yielding the result that $(\delta, -\varepsilon)$ is an a-boundary.

If $[\delta]$ is in a component of $E^2 - [\varepsilon]$ on which the index of ε is strictly positive, the above argument shows that $(\varepsilon, -\delta)$ is an a-boundary. By Lemma 5.3, $(\delta, -\varepsilon)$ is then an a-boundary. This completes the proof for Case 1.

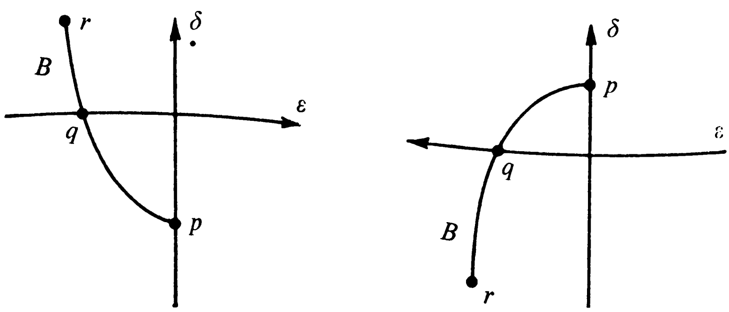

FIGURE 1

Case 2. As Figure 1 shows, there always exists an arc B with a representation ϕ satisfying the hypothesis of Lemma 5.2 when $[\delta] \cap [\varepsilon] \neq \emptyset$. That B and $[qr(\phi)]$ are interior arcs is seen immediately from [5, Lemma 2, p. 1085]. The conclusion then follows from Lemma 5.2.

REMARK. Let δ and ε be a pair of normal representations which intersect normally. It is desired to determine if (δ, ε) is an a-boundry. If both δ and $-\varepsilon$ are interior boundaries, apply Theorem 3. If δ is not an interior boundary, and $-\varepsilon$ is, test $(-\varepsilon, -\delta)$; by Lemma 5.3, this is the same as testing (δ, ε). So we may assume that $-\varepsilon$ is not an interior boundary. Then Theorem 2 can be applied to (δ, ε). If condition (1), (2a), (3a), or (3b) holds, the curve that arises can be modified to be normal and tested by the methods of [6]. If condition (2b), (3c) or (3d) holds, then ε^* and ε^{**} are modified [6, p. 55 and p. 57] to normal curves mod ε^* and mod ε^{**} with strictly less vertices than ε. For any curve ζ, (δ, ζ) is an a-boundary if and only if $(\delta, \text{mod } \zeta)$ is; ζ is an interior boundary if and only if mod ζ is. Thus Theorem 2 is repeatedly applied until either case (1), (2a), (3a), or (3b) arises or until ε is "cut" into a Jordan curve J. It is then desired to test (δ, J). If J is positively oriented, (1) of Theorem 2 applies. If J is negatively oriented, test $(-J, -\delta)$.

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